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Note

Note on packings in Grassmannian space $G(3,1)$

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How must $2N$ non-overlapping equal circles forming antipodal pairs be packed on a sphere so that the angular diameter of the circles will be as great as possible? In this note, some unnoticed putative solutions to this problem are mentioned, and attention is called to the Danzerian rigidity of the graphs of locally optimal antipodal packings.

In a recent paper, Conway et al. [1] have studied the problem of the best packing of N points in Grassmannian space $G(3,1)$, or in other words, of the best packing of N lines through the origin in $\mathbb{R}^3$. This problem is the same as to find the antipodal arrangement of $2N$ points on S^2 where the minimum distance between any two of the points is a maximum; or, that is the same, to find the packing of $2N$ non-overlapping equal circles of the largest diameter on S^2 with the condition that all circles form antipodal pairs. This problem is closely related to the well-known Tammes problem [3], that is, the problem of the densest spherical circle packing where, however, there is no central symmetry constraint. We will refer to packings in these two cases as *antipodal* and *unconstrained* packings.

Conway et al. [1] have made extensive computations and have found a number of putatively optimal antipodal packings. On their nice results, we want to make two comments.

1. It is known that Fejes Tóth [4] has solved the antipodal packing problem for $N \leqslant 6$. Conway et al. [1] have mentioned this fact, and have shown that the solution conjectured by A. Heppes and published by Fejes Tóth [4] for $N = 7$ is optimal, but have not mentioned any early solutions for $N \geqslant 8$. To our knowledge, however, interestingly enough, there are some arrangements published as conjectured solutions to the unconstrained packing problem, and since they have central symmetry, it turned out that they provide putative solutions to the antipodal packing problem, and they are in complete agreement with the results given by Conway et al. [1]. These are the cases $N = 10$ [5], $N = 11, 15$ [7], $N = 16$ [6]. Additionally we note here that the solutions for $N = 8, 12$ and one of the two different solutions for $N = 9$ were obtained independently also by us [9] using a modified version of our heating technique developed for unconstrained packings [8]. Data of these packings are compiled in table 1.

Table 1
Early results in antipodal packing of $2N$ points on S^2.

N	$\min\theta_1$	f	Reference
2	90.000000	0	[4]
3	90.000000	−4	[4]
4	70.528779	0	[4]
5	63.434948	−4	[4]
6	63.434948	−10	[4]
7	54.735610	0	[4]
8	49.639933	0	[9]
9	47.982132	−4	[9]
10	46.674620	−4	[5]
11	44.403126	0	[7]
12	41.882040	−4	[9]
15	38.134942	−4	[7]
16	37.377368	0	[6]

Proven and putative maximum values of the minimum angular distance θ_1 between points in degrees. (Optimality has been proved for $N \leqslant 6$ by Fejes Tóth [4] and for $N = 7$ by Conway et al. [1].) Non-positivity of the generic degree of freedom f of the graph of packing shows that the necessary condition of Danzerian rigidity of the graph is fulfilled.

2. Consider the circle packing version of the unconstrained packing problem. To an arrangement of non-overlapping equal circles there corresponds a *graph*. The vertices of the graph are the centres of the spherical circles, and the edges of the graph are the shorter great circle arcs joining the centres of the touching spherical circles. Thus, all edges of the graph are of equal length. Danzer [2] has considered the graph so that the edges can rotate freely around the vertices and the edge lengths can vary freely but simultaneously and in the same proportion. He has defined the graph to be *rigid* if the edge system with the mentioned properties cannot admit motions other than isometries. Danzer's idea ("almost conjecture") is: If the graph is not rigid then, in general, the packing is not locally optimal, that is, can be improved. (We note here that according to the generally used definition of rigidity of graphs, unlike Danzer's, the edge length cannot change freely.)

Let the graph of packing have v vertices and e edges. The position of v vertices on the sphere is determined by $2v$ coordinates from which 3 should be given in order to prevent rigid motion on the spherical surface. So, we have $2v - 3$ unknown coordinates, and we have an additional unknown – the edge length. Therefore, the number of unknowns is $2v - 2$. For each edge, there is an equation expressing the fact that the distance between the end points of the edge is equal to the edge length. So, we have e equations. If $e < 2v - 2$, then the position of the vertices is not uniquely determined, the graph can have a motion. Thus, a necessary condition of Danzerian rigidity of the graph is

$$e \geqslant 2v - 2. \tag{1}$$

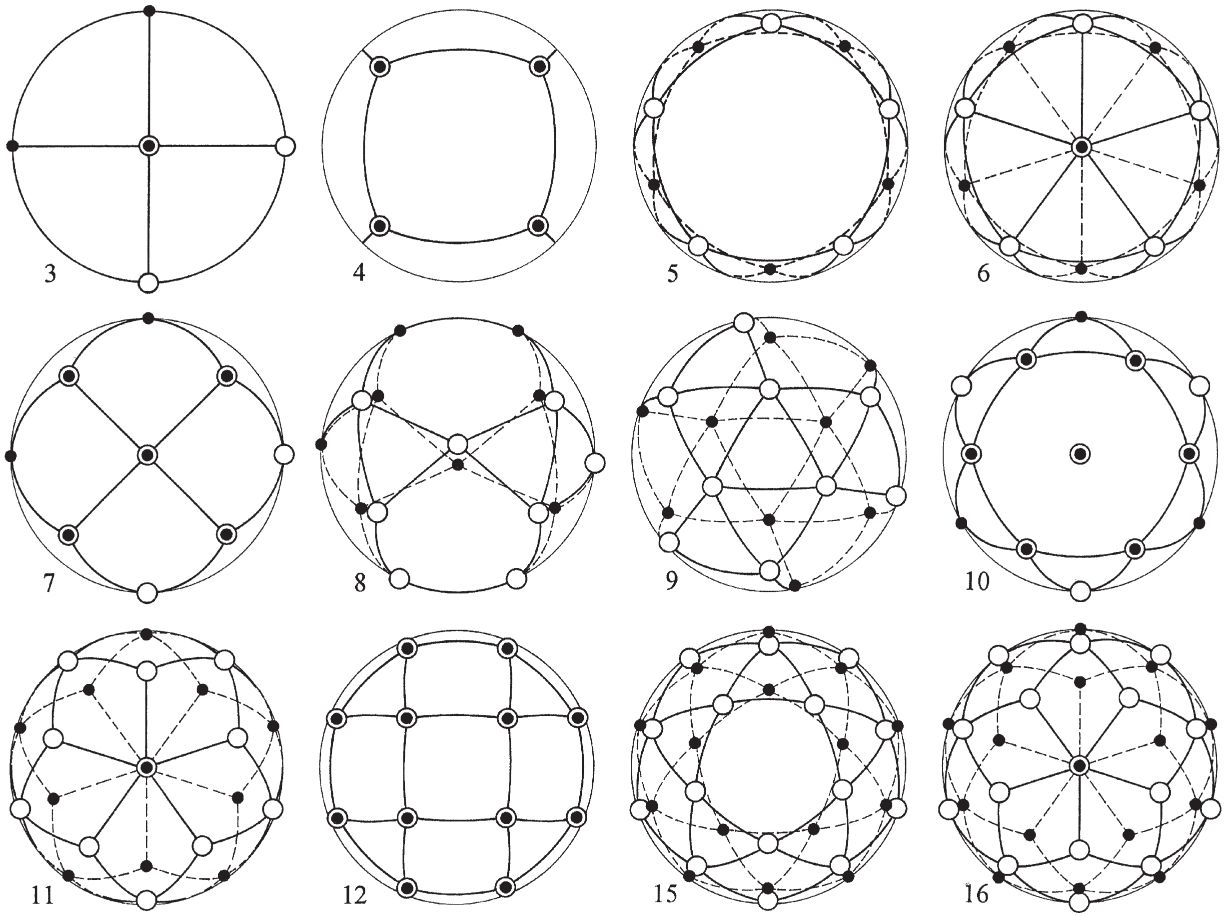

Figure 1. Graphs of the best antipodal packings for $N = 3$–12 and $15, 16$ in orthogonal projection. A point and its antipodal counterpart are marked with a small empty circle and a small black spot.

A detailed investigation of rigidity and Danzerian rigidity of the graph of spherical circle packing can be found in [8].

Consider now the antipodal packing. In this case, the definition of the graph is the same as that for unconstrained packing. Let the graph of packing have $2N$ vertices and E edges. E is even because, due to central symmetry, the edges occur in pairs. Essentially, the half of the graph can be characterized in the same way as the whole graph of the unconstrained packing, that is, taking $e = E/2$ and $v = N$ in (1), the necessary condition of Danzerian rigidity of the graph of antipodal packing is obtained:

$$E \geqslant 4N - 4. \tag{2}$$

By (2), the generic degree of freedom f can be defined as

$$f = 4N - 4 - E.$$

If $f > 0$, the graph cannot be rigid in Danzerian sense and, due to Danzer's idea, it is expected that the edge length is not a local maximum, that is, the packing can be improved. If among the vertices there are some isolated (rattling) points, when counting the edges, we reckon that 4 edges are needed to fix each antipodal pair of the isolated points. For $3 \leqslant N \leqslant 12$ and $N = 15, 16$, we have shown the graph of antipodal packing in figure 1, and have determined the generic degree of freedom f of the graph (table 1). It can be ascertained that, in all cases examined, the necessary condition of Danzerian rigidity is fulfilled. Rigidity check is a useful tool, especially for larger values of N, to select packings that are improvable. From this point of view it would be nice to know that all antipodal packings given by Conway et al. [1] are rigid in Danzerian sense.

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References

[1] J.H. Conway, R.H. Hardin and N.J.A. Sloane, Packing lines, planes, etc.: Packings in Grassmannian spaces, Experiment. Math. 5 (1996) 139–159.

[2] L. Danzer, Endliche Punktmengen auf der 2-Sphäre mit möglichts grossem Minimalabstand, Habilitationsschrift, Universität Göttingen (1963); English translation: Finite point-sets on S^2 with minimum distance as large as possible, Discrete Math. 60 (1986) 3–66.

[3] L. Fejes Tóth, *Regular Figures* (Pergamon, MacMillan, New York, 1964).

[4] L. Fejes Tóth, Distribution of points in the elliptic plane, Acta Math. Acad. Sci. Hungar. 16 (1965) 437–440.

[5] H. Rutishauser, Über Punktferteilungen auf der Kugel-fläche, Comment. Math. Helv. 17 (1945) 327–331.

[6] K. Schütte und B.L. van der Waerden, Auf welcher Kugel haben 5, 6, 7, 8 order 9 Punkte mit Mindestabstand Eins Platz?, Math. Ann. 123 (1951) 96–124.

[7] J. Strohmajer, Über die Verteilung von Punkten auf der Kugel, Ann. Univ. Sci. Budapest. Sect. Math. 6 (1963) 49–53.

[8] T. Tarnai and Zs. Gáspár, Improved packing of equal circles on a sphere and rigidity of its graph, Math. Proc. Cambridge Philos. Soc. 93 (1983) 191–218.

[9] T. Tarnai and Zs. Gáspár, Packing of equal circles in the elliptic plane and in special domains of the Euclidean plane, in: *3rd Geometry Festival. Int. Conf. on Packings, Coverings and Tilings*, Budapest (1996).